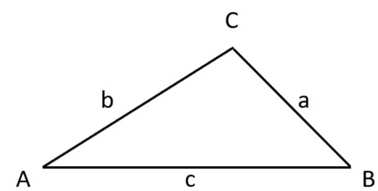
Non-right Triangle Trigonometry

Solving Non-Right Triangles: Sine and Cosine Laws

A triangle has six defining elements: three sides and three angles. To **construct** or **draw** a triangle, only three components are required if at least one of them is a side length.

In trigonometry, we use the **Sine** and **Cosine** Laws to solve or construct a non-right triangle.

Capital letters A, B, and C on the vertices are used to represent angles, and the corresponding lowercase letters a, b, and c represent the lengths of the opposite sides.



Sine Law:

$$\frac{sinA}{a} = \frac{sinB}{b} = \frac{sinC}{c}$$

Cosine Law:

$$c^2 = a^2 + b^2 - 2ab \,.\cos C$$

This formula calculates side "c". To determine the other sides, just alternate letters.

To solve, construct, or draw a triangle given three components as described, it can be done in four different ways. The first three (s - s – s), (s – A – s), and (A – A - s) are straightforward. Although the fourth case (s – s -A) is more complex and can be divided into 4 subcases. All possible configurations are summarized on the following pages, accompanied by detailed explanations, examples, and solutions.

Case 1. Three sides known as (s – s - s) a, b and c, are given. Find angles A, B, and C.

Rearranging the "Cosine law", we can find any angles, like:

$$\cos A = \frac{b^2+c^2-a^2}{2bC} \qquad \text{Then:} \qquad A = cos^{-1}(\frac{b^2+c^2-a^2}{2bC})$$

The second angle can be calculated the same way, or by using the "Sine law":

$$\sin B = \frac{b \, . \sin A}{a} \qquad\qquad B = sin^{-1}(\frac{b \, . \sin A}{a})$$

The third angle: C = 180° – (A + B). There will be only one possible solution.

Case 2. Given 2 sides "a, and b" and the included angle C, known as (s – A - s).

To find the side c, use the cosine law:

$$c^2 = a^2 + b^2 - 2ab \, . \cos C$$

Then calculate another angle, like B, by using the sine law:

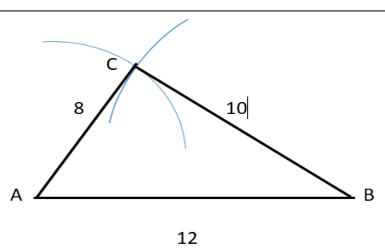
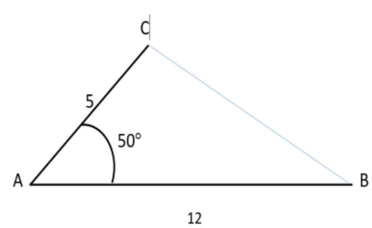
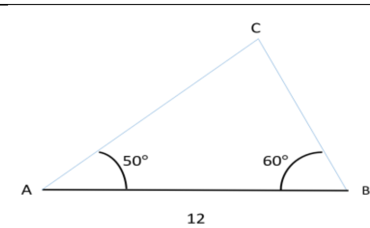
$$\sin B = \frac{b \, . \sin C}{c} \qquad \text{Therefor:} \qquad B = sin^{-1}(\frac{b \, . \sin C}{c})$$

The third angle B = 180° – (A + C). Only one possible solution.

These two cases can only be solved using the Cosine Law as a starting point.

Case 3. Two angles and one side (A – s - A, and A – A - s) are given. In both cases, the third angle can be found by using the formula A + B + C = 180°, so the two cases are the same. Use the sine law to find the two other sides. There is only one possible solution.

Examples of constructing a triangle for the first three cases are summarized in the following diagrams:

Case 1: (s – s - s) Ex: a = 10, b = 8, c = 12 cm	**Case 2:** (s – A - s) Ex: b = 5 cm, A = 50°, c = 12 cm	**Case 3:** (A – s - A) Ex: A = 50°, c =12 cm, B = 60°
		

Solutions for Cases 1, 2, and 3 examples:

Ex 1: $\cos A = \frac{12^2+8^2-10^2}{2x12x8} = \frac{108}{192} = 0.56$ Therefore: $A = \cos^{-1} 0.56 = 55.77°$

The same way: $\cos B = \frac{12^2+10^2-8^2}{2x12x10} = \frac{180}{240} = 0.75$ Therefore: $B = os^{-1} 0.75 = 41.41°$

Or we can use the sine law to find the angle B, then: $C = 180 - 41.41 - 55.77 = 82.82°$

Ex 2: $a^2 = 12^2 + 5^2 - 2x12x5.\cos(50°) = 91.87$ Then: $a = 9.58$ cm

Use the sine law to find the angle B: $\sin B = \frac{b.\ \sin A}{a} = \frac{5.\ \sin(50°)}{9.58} = 0.40$ Then: $B = 23.57°$

And $C = 180 - 50 - 23.57 = 106.43°$

Ex 3: We need to find the angle C first to have one angle and the opposite side to apply the Sine law:

$$C = 180 - 50 - 60 = 70°$$

Now use the sine law to find sides a and b:

$$\frac{sinA}{a} = \frac{sinC}{c} \quad \text{Then:} \quad a = \frac{c.\sin A}{\sin C} = \frac{12.\sin 50}{\sin 70} = 9.78\ cm$$

$$\frac{sinB}{b} = \frac{sinC}{c} \quad \text{Then:} \quad b = \frac{c.\sin B}{\sin C} = \frac{12.\sin 60}{\sin 70} = 11.06\ cm$$

* * *

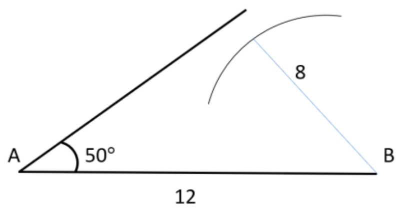
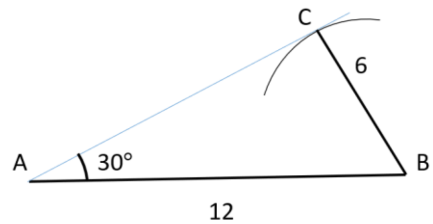
Case 4. Given two sides and one angle opposite to one of them, like sides a and c, and the angle A, which is known as (A - s – s). The task is to find angles B and C, and then side b.

First, by using the sine law, find angle C:

$$\sin C = \frac{c.\ \sin A}{a} \quad \text{Therefor:} \quad C = sin^{-1}\left(\frac{c.\ \sin A}{a}\right)$$

This case is more complicated than the previous three cases; here again, 4 scenarios are possible:

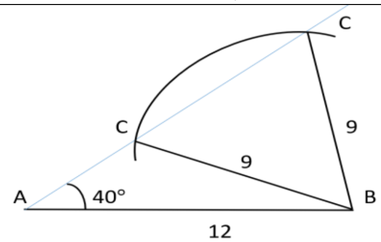
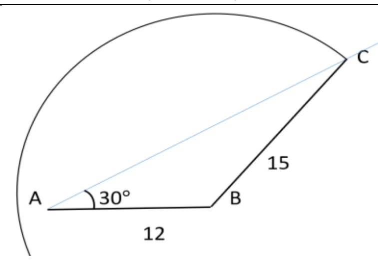
Examples of solving a triangle for the first two scenarios are summarized in the following diagrams:

Case 4 – 1: (A - s – s) Ex: A = 50°, a = 8, c = 12 cm	**Case 4 – 2:** (A - s – s) Ex: A = 30°, a = 6, c = 12 cm
	

1) It is possible $\frac{c\,.\,\sin A}{a} > 1$, or (sin C > 1) , this is wrong because: -1 $\leq sin\theta \leq 1$, So we can conclude <u>there is no solution</u>. Geometrically, this means that the side "a" is too short to reach the side b.

2) If $\frac{c\,.\,\sin A}{a} = 1$, means sin C = 1, therefore C = 90°, <u>there is one solution, a right triangle.</u>

Examples of solving a triangle for the next two scenarios are summarized in the following diagrams:

Case 4 – 3: (A - s – s) Ex: A = 40° , a = 9, c = 12 cm	**Case 4 – 4:** (A - s – s) Ex: A = 30°, a = 15, c = 12 cm
	

3) In this case; $\sin C = \frac{c\,.\,\sin A}{a} < 1$, which means sin C < 1. This corresponds to the **ambiguous** (s - s – A) case, in which **two solutions** may exist (when the geometry permits, only if a < c). For a given sine value, there are two possible angles in the first and second quadrants, C_1 and C_2, therefore 2 different angles, B and sides b, thus 2 different triangles.

$C_1 = \sin^{-1}\left(\frac{c\,.\,\sin A}{a}\right)$, an acute angle, and then; $C_2 = 180° – C_1$ will be an obtuse angle.

$B_1 = 180 - A - C_1$, and $B_2 = 180 - A - C_2$.

Then calculate the side c for each pair of angles B and C using: $c = \frac{b \sin C}{\sin B}$

4) Again $\frac{c\,.\,\sin A}{a} < 1$, and $C = \sin^{-1}\left(\frac{c\,.\,\sin A}{a}\right)$, or sin C < 1, but a >c. In this case, the second angle, C2, is too large, which means $C_2 + B > 180°$. As shown in the geometric diagram, the circle with the center B and the radius "a" can cut side "c" at only one point; the other point falls outside, therefore, <u>there is only one solution</u>, which is: $c = \frac{b \sin C}{\sin B}$

Solutions for Cases 4 examples

Ex 1: $a = 8$ cm, $c = 12\ cm$, and $A = 50°$. Therefore:

$\sin C = \frac{12\ .\ \sin 50}{8} = 1.15$, means sin C > 1, therefore no solution.

Ex 2: $a = 6$ cm, $c = 12\ cm$, and $A = 30°$. Therefore:

$\sin C = \frac{12\ .\ \sin 30}{6} = 1$, means sin C = 1, therefore C = 90°, there is one solution, a right triangle.

Therefore: $B = 90 - A = 60°$. Side b can be calculated by any of those 2 ways or by the Pythagorean formula: $b^2 = c^2 - a^2$

Ex 3: $a = 9$ cm, $c = 12\ cm$, and $A = 40°$. Therefore: $\sin C = \frac{12\ .\ \sin 40}{9} = 0.86$, then $C_1 = 59°$ and $C_2 = 180 - 59 = 121°$. Therefore, there are 2 angles B and 2 sides b:

$B_1 = 180 - 40 - 59 = 81°$, $b_1 = 13.9\ cm$ and $B_2 = 180 - 40 - 121 = 19°, b_1 = 4.9\ cm$

Ex 4: $a = 15$ cm, $c = 12\ cm$, and $A = 30°$. Therefore: $\sin C = \frac{12\ .\ \sin 30}{15} = 0.4$, then $C_1 = 23.6°$

$C_2 = 180 - 23.6 = 156.4$, So, no second angle is possible. Then $b = 24.1\ cm$ Only one solution.

As predicted, since a>c, the second angle falls outside of side b; so, there is no second triangle.